

The OPEs of Spin-4 Casimir Currents in the Holographic $SO(N)$ Coset Minimal Models

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Abstract

We compute the operator product expansion (OPE) between the spin-4 current and itself in the WD_4 coset minimal model with $SO(8)$ current algebra. The right hand side of this OPE contains the spin-6 Casimir current which is also a generator of WD_4 coset minimal model. Based on this $N = 8$ result, we generalize the above OPE for the general N (in the $WD_{\frac{N}{2}}$ coset minimal model) by using two N -generalized coupling constants initiated by Hornfeck sometime ago: the simplest OPE for the lowest higher spin currents. We also analyze the similar OPE in the WB_3 (and $WB_{\frac{N-1}{2}}$) coset minimal model with $SO(7)$ current algebra. The large N 't Hooft limits are discussed. Our results in two dimensional conformal field theory provide the asymptotic symmetry, at the quantum level, of the higher spin AdS_3 gravity found by Chen et al.

1 Introduction

In the Gaberdiel and Gopakumar proposal [1, 2], the WA_{N-1} minimal model conformal field theory is dual, in the 't Hooft $\frac{1}{N}$ expansion, to the higher spin theory of Vasiliev on the AdS_3 coupled to one complex scalar. The higher spin gauge fields in the bulk AdS_3 couple to a conserved higher spin currents (whose charges form an extended global symmetry of the conformal field theory) in the boundary theory. See the recent review papers [3, 4].

The $SU(N)$ spin-3 Casimir construction in the WA_{N-1} minimal model (described in terms of a coset) has been found in [5]. The four independent cubic terms are made of the spin-1 currents in the two factors in the numerator of the coset. Then the OPE between the spin-3 current and itself generates the spin-4 current [6] which consists of quartic terms and quadratic terms with derivatives in terms of above spin-1 currents. Furthermore, the $SO(N)$ spin-4 Casimir construction in the $WD_{\frac{N}{2}}$ and $WB_{\frac{N-1}{2}}$ minimal models [7] is found in [8] with the observation of [9, 10]. The quartic terms, cubic terms with one derivative and quadratic terms with two derivatives in the spin-4 current (with two unknown coefficient functions) are given in terms of the spin-1 currents in the two factors in the numerator of the coset. It is natural to ask what happens when one computes the OPE between the spin-4 current and itself.

In this paper, we compute the OPE between the spin-4 Casimir current and itself, the simplest OPE for the lowest higher spin currents, in the $WD_{\frac{N}{2}}$ coset minimal model with $N = 8$. During this computation, the spin-6 Casimir current arises on the right hand side of this OPE and the two unknown coefficients in the spin-4 current are fixed. We would like to generalize the above OPE for the general N by using two N -generalization coupling constants [11]. We also analyze the similar OPE in the $WB_{\frac{N-1}{2}}$ coset minimal model.

As described in [11], the findings over there did not answer for what kinds of the field contents for WD_4 (or WB_3) algebra are present. In this paper, one sees the field contents for the higher spin currents, explicitly, living in the specific coset model we are considering. In [12], they started with the most general ansatz for the OPEs between the spin-4, spin-6, spin-8 currents and determined the various structure constants using the Jacobi identities between these higher spin currents. This can be classified by the approach 1 in the context of [13]. With the identification of the above algebra as Drinfeld-Sokolov reduction (can be described as the approach 2 in [13]) of higher spin algebra, they applied to the coset model by comparing the central charge and the self-coupling constant of the spin-4 current. However, it is not clear how one can see the explicit form for the higher spin currents on the coset model. Since the approach 3 in [13] we are using is based on the specific model and the

higher spin currents are made of the fields in the coset model, one can analyze the zero mode eigenvalue equation which is necessary to describe the three-point function with real scalar as in [14, 15, 8]. Note that the zero modes satisfy the commutation relations of the underlying finite dimensional Lie algebra.

According to the result of [16, 13], one expects that the additional currents as well as WA_{N-1} (or $WD_{\frac{N}{2}}$) currents with arbitrary levels appear. As one puts one of the levels as one, then this extra current in the OPE disappears completely. In the spirit of [3, 17], one can think of more general algebra rather than conventional WA_{N-1} (or $WD_{\frac{N}{2}}$) algebra: the existence of additional higher spin currents. One of the levels in this general coset model is not equal to 1. Then one cannot use the isomorphism between the coset construction and the Drinfeld-Sokolov reduction, explained in [12], because this isomorphism restricts to have one of the levels as 1 [13]. This implies that for this general coset model with levels (k, l) , one cannot follow the procedure developed in [12] directly¹. Our presentation in this paper will give some hints in order to describe the general coset model with levels (k, l) in two dimensional CFT.

In section 2, we review the diagonal coset minimal models and describe the spin-2 current with the central charge.

In section 3, by taking the OPE between the spin-4 currents, we construct the spin-6 current in the $WD_{\frac{N}{2}}$ minimal model with $N = 8$. By realizing the presence of two structure constants which depend on the general N , the OPE between the spin-4 currents which is valid for any N is given.

In section 4, we describe the similar OPE in the $WB_{\frac{N-1}{2}}$ minimal model.

In section 5, we summarize what we have found and comment on future directions.

In the Appendices, for convenience, we present the relative coefficient functions in terms of two undetermined ones for each minimal model. These appeared in [8] previously.

During this preparation, we have noticed that the work of [12] has some overlaps with our work although they did not consider the explicit realizations we are using. Furthermore, we

¹ In other words, the direct construction using the Jacobi identity (without knowing the realization of the higher spin currents) is not connected to the Casimir (coset) construction directly. This implies that even though one has the self-coupling constant for the spin-4 current from the work of [11, 12], the coset construction itself (a realization in the specific coset model) is interesting in order to identify a complete set of generating currents which will have larger symmetry. Once we determine the lower higher spin currents using the coset construction explicitly, then the next undetermined higher spin currents can be generated, in principle, by computing the OPEs between the known higher spin currents repeatedly (i.e., by analyzing the various singular terms). In general, this procedure will be quite involved. Contrary to the approach 1 (which can be done only if one knows the number of currents with given spins), the resulting extended algebra via the approach 3 is associative by construction and therefore we do not have to check the Jacobi identities separately. It would be interesting to study this more general coset model and its AdS_3 gravity dual which will be beyond the scope of this paper.

also have noticed the presence of a previous work by Hornfeck [11] independently before their paper came out.

We are heavily using the package by Thielemans [18].

2 The GKO coset construction: Review

For the diagonal coset model

$$\frac{G}{H} = \frac{\widehat{SO}(N)_k \oplus \widehat{SO}(N)_1}{\widehat{SO}(N)_{k+1}}, \quad (2.1)$$

the spin-1 fields, $J^{ab}(z)$ with level 1 and $K^{ab}(z)$ with level k , generate the affine Lie algebra G . The indices $a, b = 1, 2, \dots, N$ are in the representation of finite dimensional Lie algebra $SO(N)$. Their OPEs [19] are

$$\begin{aligned} J^{ab}(z)J^{cd}(w) &= -\frac{1}{(z-w)^2} (-\delta^{bc}\delta^{ad} + \delta^{ac}\delta^{bd}) \\ &+ \frac{1}{(z-w)} \left[\delta^{bc}J^{ad}(w) + \delta^{ad}J^{bc}(w) - \delta^{ac}J^{bd}(w) - \delta^{bd}J^{ac}(w) \right] + \dots, \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} K^{ab}(z)K^{cd}(w) &= -\frac{1}{(z-w)^2} k(-\delta^{bc}\delta^{ad} + \delta^{ac}\delta^{bd}) \\ &+ \frac{1}{(z-w)} \left[\delta^{bc}K^{ad}(w) + \delta^{ad}K^{bc}(w) - \delta^{ac}K^{bd}(w) - \delta^{bd}K^{ac}(w) \right] + \dots. \end{aligned} \quad (2.3)$$

The spin-1 field, $J'^{ab}(z)$ with level $(k+1)$, that generates the affine Lie subalgebra H , is given by

$$J'^{ab}(z) = J^{ab}(z) + K^{ab}(z). \quad (2.4)$$

The Sugawara stress energy tensor for the coset (2.1) is, with (2.4),

$$\begin{aligned} T(z) &= -\frac{1}{4(N-1)} (J^{ab}J^{ab})(z) - \frac{1}{4(k+N-2)} (K^{ab}K^{ab})(z) \\ &+ \frac{1}{4(k+N-1)} (J'^{ab}J'^{ab})(z). \end{aligned} \quad (2.5)$$

The OPE between the spin-2 currents (2.5) is given by

$$T(z)T(w) = \frac{1}{(z-w)^4} \frac{c}{2} + \frac{1}{(z-w)^2} 2T(w) + \frac{1}{(z-w)} \partial T(w) + \dots \quad (2.6)$$

The central charge in the highest singular term in (2.6) is given by

$$c = \frac{N}{2} \left[1 - \frac{(N-2)(N-1)}{(N+k-2)(N+k-1)} \right] \leq \frac{N}{2}. \quad (2.7)$$

The higher spin Casimir currents of spin 4 in $WD_{\frac{N}{2}}$ and $WB_{\frac{N-1}{2}}$ minimal models were constructed in [8]. In next sections, we will compute the OPEs between these spin-4 currents in each minimal model.

3 The OPE between the spin-4 current and itself in $WD_{\frac{N}{2}}$ minimal model with even N

In [8], the spin-4 current is determined, with two unknown coefficient functions, by

$$\begin{aligned} V(z) = & c_3 J^{cd} J^{ef} K^{cd} K^{ef}(z) + c_8 J^{cd} J^{ef} K^{ce} K^{df}(z) + c_9 J^{cd} K^{ef} K^{ce} K^{df}(z) \\ & + c_{10} K^{cd} K^{ef} K^{ce} K^{df}(z) + c_{11} J^{cd} J^{cd} J^{ef} J^{ef}(z) + c_{12} J^{cd} J^{cd} J^{ef} K^{ef}(z) + c_{13} J^{cd} J^{cd} K^{ef} K^{ef}(z) \\ & + c_{14} J^{cd} K^{cd} K^{ef} K^{ef}(z) + c_{15} K^{cd} K^{cd} K^{ef} K^{ef}(z) + c_{18} J^{cd} J^{ce} K^{ef} K^{df}(z) + c_{21} J^{cd} J^{ce} J^{df} J^{ef}(z) \\ & + c_{22} J^{cd} J^{ce} J^{df} K^{ef}(z) + d_1 \partial J^{ab} \partial J^{ab}(z) + d_2 \partial^2 J^{ab} J^{ab}(z) + d_3 \partial K^{ab} \partial K^{ab}(z) + d_4 \partial^2 K^{ab} K^{ab}(z) \\ & + d_5 \partial^2 J^{ab} K^{ab}(z) + d_6 \partial J^{ab} \partial K^{ab}(z) + d_7 J^{ab} \partial^2 K^{ab}(z) + d_8 J^{ab} \partial J^{ac} K^{bc}(z) \\ & + d_9 J^{ab} K^{ac} \partial K^{bc}(z), \end{aligned} \quad (3.1)$$

where the coefficient functions are given in (A.1). Let us first consider the particular $N = 8$ case in the $WD_{\frac{N}{2}}$ minimal model². It is straightforward to compute the OPE between this spin-4 current (3.1) and itself for given the basic OPEs from (2.2) and (2.3). We present the final result and then describe the details we use

$$\begin{aligned} V(z) V(w) = & \frac{1}{(z-w)^8} \frac{c}{4} + \frac{1}{(z-w)^6} 2T(w) + \frac{1}{(z-w)^5} \frac{1}{2} 2\partial T(w) \\ & + \frac{1}{(z-w)^4} \left[\frac{3}{20} 2\partial^2 T + \frac{42}{(5c+22)} \left(T^2 - \frac{3}{10} \partial^2 T \right) + C_{44}^4 V \right] (w) \end{aligned}$$

² In this case, the currents are given by the spin-4, the spin-6 currents as well as the spin-2 current. Moreover, there exists one additional spin-4 current. For general N , this last current has a spin- $\frac{N}{2}$. This implies that for large N behavior, the OPEs between the lower higher spins in the $WD_{\frac{N}{2}}$ minimal model do not contain this spin- $\frac{N}{2}$ field. However, for WD_4 minimal model, we will see that the extra spin-4 current does not appear in the OPE between the other spin-4 currents. This can be explained by the existence of the outer \mathbf{Z}_2 automorphism [20, 12] under which the spin-2, spin-4 and spin-6 currents are even while the extra spin-4 current is odd. For general N , the 'orbifold' subalgebra of $WD_{\frac{N}{2}}$ [21] is generated by the quadratic term in the extra spin- $\frac{N}{2}$ (of spin N) as well as its higher derivative terms, in addition to the above spin-2, spin-4, \dots , spin- $(N-2)$.

$$\begin{aligned}
& + \frac{1}{(z-w)^3} \left[\frac{1}{30} 2 \partial^3 T + \frac{1}{2} \frac{42}{(5c+22)} \partial \left(T^2 - \frac{3}{10} \partial^2 T \right) + \frac{1}{2} C_{44}^4 \partial V \right] (w) \\
& + \frac{1}{(z-w)^2} \left[\frac{1}{168} 2 \partial^4 T + \frac{5}{36} \frac{42}{(5c+22)} \partial^2 \left(T^2 - \frac{3}{10} \partial^2 T \right) + \frac{5}{36} C_{44}^4 \partial^2 V \right. \\
& + \frac{24(72c+13)}{(5c+22)(2c-1)(7c+68)} \left(T(T^2 - \frac{3}{10} \partial^2 T) - \frac{3}{5} \partial^2 TT + \frac{1}{70} \partial^4 T \right) \\
& - \frac{(95c^2+1254c-10904)}{6(5c+22)(2c-1)(7c+68)} \left(\frac{1}{2} \partial^2 (T^2 - \frac{3}{10} \partial^2 T) - \frac{9}{5} \partial^2 TT + \frac{3}{70} \partial^4 T \right) \\
& \left. + \frac{28}{3(c+24)} C_{44}^4 \left(TV - \frac{1}{6} \partial^2 V \right) + C_{44}^6 W \right] (w) \\
& + \frac{1}{(z-w)} \left[\frac{1}{1120} 2 \partial^5 T + \frac{1}{36} \frac{42}{(5c+22)} \partial^3 \left(T^2 - \frac{3}{10} \partial^2 T \right) + \frac{1}{36} C_{44}^4 \partial^3 V \right. \\
& + \frac{1}{2} \frac{24(72c+13)}{(5c+22)(2c-1)(7c+68)} \partial \left(T(T^2 - \frac{3}{10} \partial^2 T) - \frac{3}{5} \partial^2 TT + \frac{1}{70} \partial^4 T \right) \\
& - \frac{1}{2} \frac{(95c^2+1254c-10904)}{6(5c+22)(2c-1)(7c+68)} \partial \left(\frac{1}{2} \partial^2 (T^2 - \frac{3}{10} \partial^2 T) - \frac{9}{5} \partial^2 TT + \frac{3}{70} \partial^4 T \right) \\
& \left. + \frac{1}{2} \frac{28}{3(c+24)} C_{44}^4 \partial \left(TV - \frac{1}{6} \partial^2 V \right) + \frac{1}{2} C_{44}^6 \partial W \right] (w) + \dots
\end{aligned} \tag{3.2}$$

Let us first consider the highest singular term. From the explicit structure of the eighth-order pole, one obtains the following expression

$$192k(2+k)(4+k)(6+k)(7+k)(9+k)(11+k)(13+k) c_{10}^2. \tag{3.3}$$

By normalizing that this expression (3.3) is equal to $\frac{c}{4}$ with

$$c_{N=8} = \frac{4k(13+k)}{(6+k)(7+k)} \tag{3.4}$$

coming from the central charge (2.7), one determines the unknown coefficient c_{10} as follows:

$$c_{10}^2 = \frac{1}{192(2+k)(4+k)(6+k)^2(7+k)^2(9+k)(11+k)} = \frac{(-4+c)^4}{130056192(24+c)(22+5c)}. \tag{3.5}$$

We also express the coefficient c_{10} in terms of the central charge (3.4) for convenience. Therefore, the central term is given by $\frac{c}{4}$ as in (3.2).

The next singular term, seventh pole, does not have any nonzero fields. In the sixth order pole, there exists a spin-2 current $T(w)$ with coefficient 2 which is a structure constant between the spin-4, the spin-4 and the spin-2 currents. In the fifth order pole, one obtains the descendant field $\partial T(w)$ of $T(w)$ with the coefficient function $\frac{1}{2} \times 2 = 1$ where the relative coefficient function $\frac{1}{2}$ is known from the spins of the current $V(z)$, the current $V(w)$, the current $T(w)$ and the number of derivatives in the descendant field [22, 23, 13, 17].

In the fourth order pole, one also has a descendant field originating from the spin-2 current $T(w)$ appeared in the higher-order singular term with known coefficient $\frac{3}{20} \times 2 = \frac{3}{10}$. Furthermore, one expects that there exist a spin-4 quasi primary field as well as the spin-4 primary field $V(w)$ by remembering the OPE of spin-4 currents in the extended conformal algebra [24, 25, 26, 23, 27], denoted by $\mathcal{W}(2, 4)$ along the line of [13], where the higher spin current is of spin-4³. Then it turns out that the fourth-order pole is given by the one in (3.2) and the self-coupling constant for the spin-4 is given by

$$(C_{44}^4)^2 = \frac{12(4+k)(9+k)}{(2+k)(11+k)} = \frac{18(c+24)}{(5c+22)}, \quad (3.6)$$

where we also write down C_{44}^4 in terms of the central charge. One can also determine the remaining unknown coefficient function c_8 as follows:

$$c_8 = \frac{(6+k)}{105} \left[(7+k) \sqrt{(4+k)(9+k)(148+k(13+k))} + (4+k)(133+k(16+k)) \right] c_{10} \quad (3.7)$$

with (3.5). Of course, the structure constant (3.6) is different from the corresponding one in the $\mathcal{W}(2, 4)$ algebra as in [13]. For the WD_4 algebra, the field contents are given by the spin-4, the spin-4 and the spin-6 currents as in footnote 2 while $\mathcal{W}(2, 4)$ algebra contains only the spin-4 current. The above structure constant (3.6) coincides with the general- N dependent structure constant found by Hornfeck by substituting $N = 8$ with

$$\begin{aligned} (C_{44}^4)^2 &= \frac{n}{d}, \\ n &\equiv 18 \left[2c^2(-18 + (-2 + N)N) + 2N(-28 + N(5 + 6N)) \right. \\ &\quad \left. + 3c(-8 + N(80 + N(-49 + 6N))) \right]^2, \\ d &\equiv (22 + 5c)(c + (-5 + c)N + 4N^2)(c(-4 + N)(-3 + N) + N(-5 + 2N)) \\ &\quad \times (2c(2 + N) + (-4 + N)(-2 + 3N)). \end{aligned} \quad (3.8)$$

Recently, this expression is reproduced in [12].

In the third-order pole, there are no additional spin-5 quasi-primary fields. There are one descendant field coming from $T(w)$ and two descendant fields coming from spin-4 quasi primary and primary fields.

³If one does not know the field contents in the fourth-order pole, one can follow the method done in [17]. One performs the OPE between the spin-2 current $T(z)$ and the fourth-order pole subtracted by $\frac{3}{10}\partial^2 T(w)$ and focus on the fourth-order pole. Then one has nonzero $T(w)$ term on the right hand side of OPE. This indicates that one can consider the extra quasi-primary field containing $T^2(w)$ term with derivative term because the OPE between $T(z)$ and $T^2(w)$ provides a term $T(w)$ in the fourth order pole. The coefficient $-\frac{3}{10}$ in the derivative term can be checked via the vanishing of third-order pole in the OPE with $T(z)$. Then it is simple to compute the OPE between $T(z)$ and the fourth-order pole subtracted by $\frac{3}{10}\partial^2 T(w) + c_1(T^2 - \frac{3}{10}\partial^2 T)(w)$. Once again, the primary field condition fixes the constant as $c_1 = \frac{42}{(5c+22)}$. Of course, one should compute the OPE between $T(z)$ and $\partial^2 T(w)$ explicitly in order to obtain this result.

Now we are ready to consider the next second-order singular term ⁴. One expects that there should be a spin-6 primary field which is a generator of WD_4 minimal model. The first line of the second-order pole in (3.2) is known and the remaining three terms are characterized by three spin-6 quasi-primary fields ⁵. Although the V -independent terms are characterized by $\partial^4 T(w)$, $\partial^2 TT(w)$, $T^3(w)$ and $\partial T \partial T(w)$ (these are all possible spin-6 fields coming from the spin-2 current $T(w)$), it is very important to split two descendant terms and two quasi-primary fields in order to find out the new quasi-primary fields for given pole. These also arise in the $\mathcal{W}(2, 4)$ extended conformal algebra. Then finally, we are left with a spin-6 primary field where

$$C_{44}^6 W(z) = \text{coeff}(k) J^{12} J^{12} J^{12} J^{12} J^{12} J^{12}(z) + \dots, \quad (3.9)$$

and $\text{coeff}(k)$ is a complicated function of k and we do not present it here.

In order to determine the normalization for the spin-6 current, one should compute the highest singular term from the OPE between the current (3.9) and itself. However, it is not possible, at the moment, to do this because the spin-6 current has too many terms. By demanding that this central term should be equal to $\frac{c}{6}$, one expects that one should obtain the normalization factor as follows:

$$(C_{44}^6)^2 = \frac{6(-1+k)(14+k)(24+5k)(41+5k)}{(-6+k(13+k))(119+4k(13+k))} = \frac{12(c-1)(11c+656)}{(2c-1)(7c+68)}. \quad (3.10)$$

This structure constant (3.10) is taken from the more general N dependent expression found by Hornfeck

$$(C_{44}^6)^2 = \frac{n}{d},$$

⁴It took several months to compute the complete pole structures (up to second-order pole) with several personal computers. Although we have not checked the first-order pole explicitly, we expect that the first-order pole in (3.2) is correct. The point is whether there exists an extra quasi-primary field of spin-7 or not. Since $V(z) V(w) = V(w) V(z)$, one can reverse the arguments z and w in (3.2) with the insertion of some quasi-primary field of spin-7 and use Taylor expansions about the coordinate w . Then we have an explicit expression as in the Appendix of [16]. All the higher order terms greater than order-1 can appear as the derivative terms at the first-order pole. It turns out that the first-order term in $V(w) V(z)$ appears as the first-order pole term in (3.2) and above spin-7 field with opposite sign. Therefore, one realizes that by comparing both sides, the above quasi-primary field of spin-7 vanishes.

⁵In [13], the expression for Ω_{BS} in (5.11) is not right. The correct one is $\Omega_{BS}(w) = T(T^2 - \frac{3}{10}\partial^2 T)(w) - \frac{3}{5}\partial^2 TT(w) + \frac{1}{70}\partial^4 T(w)$ as in (3.2). Also note that the notation for the normal ordering we are using here is different from the one in [23] as emphasized in [17]. Sometimes there are several ways to express the quasi-primary field by using the identities $T\partial^2 T(z) = \partial^2 TT(z) + \frac{1}{6}\partial^4 T(z)$ and $\partial^2(T^2 - \frac{3}{10}\partial^2 T)(z) = 2\partial T \partial T(z) + 2\partial^2 TT(z) - \frac{2}{15}\partial^4 T(z)$. That is, $H_{BS}(z)$ corresponds to $\Omega(z)$ of [23], $P_{BS}(z)$ corresponds to $-\frac{5}{9}\Gamma(z)$, and $(\Omega_{BS} - \frac{1}{3}P_{BS})(z)$ corresponds to $\Delta(z)$. Note that any linear combinations of quasi-primary fields for given spin provide a different quasi-primary field. The convention for the quasi-primary fields in [26] is the same as the one in [13] while the convention for the same quantity in [23] is the same as those in [12, 25].

$$\begin{aligned}
n &\equiv 12(-1+c)(22+5c)^2(c(-6+N)(-5+N)+2N(-7+2N)) \\
&\times (2c(4+N)+(-8+3N)(-4+5N))(c(3+N)+2N(-7+6N)), \\
d &\equiv (24+c)(-1+2c)(68+7c)(c+(-5+c)N+4N^2) \\
&\times (c(-4+N)(-3+N)+N(-5+2N))(2c(2+N)+(-4+N)(-2+3N)).
\end{aligned} \tag{3.11}$$

As pointed out by Hornfeck [11], there exists also an extended conformal algebra $\mathcal{W}(2, 4, 6)$ in [27] which is nothing to do with the present case but is related to the next example WB_1 coset minimal model. By substituting $N = 3$ into the formula (3.8) and (3.11), the author could obtain the structure constants in [27] exactly. Note that the spin-4 and spin-6 currents are made of stress energy tensor and its $\mathcal{N} = 1$ superpartner of spin $\frac{3}{2}$ [13]. The explicit form is given in [28]. Furthermore, the author of [11] checked that the classical $c \rightarrow \infty$ limit of (3.8) coincides with the one in [29] where the structure constant can be obtained from the one from WA_{N-2} minimal model ⁶. Note that there exists a factor $(c-1)$ in the structure constant in (3.11). In other words, $c = 1$ implies that $k = 1$ or $k = 2(1-N)$. Then, for $k = 1$, the structure constant C_{44}^6 vanishes. This is kind of ‘minimal’ extension of conformal algebra [17] where the only higher spin current is of spin 4 while the higher spin current of spin-6 vanishes.

We claim that the lowest OPE between the spin-4 current in the $WD_{\frac{N}{2}}$ coset minimal model is characterized by (3.2) where the central charge is given by (2.7), the spin-4 current is given by (3.1), the spin-2 current is given by (2.5), the structure constant C_{44}^4 is given by (3.8), and the structure constant C_{44}^6 is given by (3.11). For the spin-4 and spin-6 currents, we have found for particular $N = 8$ case in the $WD_{\frac{N}{2}}$ minimal model (and $N = 6$ case) ⁷. For the spin-4 current at general N , there exist two unknown coefficient functions as we explained before. It would be interesting to obtain this spin-6 current which holds for arbitrary N . This can be done by computing the OPEs between the spin-4 currents (3.1) by hand.

The large (N, k) ’t Hooft limit provides the following limiting value for the central charge [1, 9, 10]

$$c \rightarrow \frac{N}{2}(1 - \lambda^2), \quad \lambda \equiv \frac{N}{N + k - 2}. \tag{3.12}$$

⁶ The self-coupling structure constant between the spin-4 currents in the WA_{N-2} minimal model is given by [30, 2] $(C_{44}^4)^2 = \frac{36(-24-48c-18c^2+224N+204cN-2c^2N-130N^2-129cN^2+c^2N^2+12N^3+18cN^3)^2}{(2+c)(22+5c)(-4+N)(-3+N)(2+c-7N+cN+3N^2)(4+2c-18N+cN+8N^2)}$. By taking the $c \rightarrow \infty$ limit, this leads to $\frac{36(-18-2N+N^2)^2}{5(-4+N)(-3+N)(1+N)(2+N)}$ which is equal to the corresponding limit of (3.8) [29].

⁷We have checked the OPE (3.2) when $N = 6$ and realize that the result is exactly the same as (3.2) by replacing the central charge $c_{N=6} = \frac{3k(9+k)}{(4+k)(5+k)}$ and the $N = 6$ for the currents and structure constants. The field contents of $WD_{\frac{N}{2}}$ minimal model are given by the spins $2, 4, \dots, (N-2), \frac{N}{2}$. The naive field contents for $N = 6$ are given by spins 2, 3, and 4 in this formula. According to the observation of [31], the minimum value of N for the above field contents in the $WD_{\frac{N}{2}}$ minimal model is equal to 8.

Furthermore, the limiting values for structure constants are obtained and they are

$$\begin{aligned}(C_{44}^4)^2 &\rightarrow \frac{36(-19 + \lambda^2)^2}{5(-3 + \lambda)(-2 + \lambda)(2 + \lambda)(3 + \lambda)}, \\(C_{44}^6)^2 &\rightarrow \frac{150(-5 + \lambda)(-4 + \lambda)(4 + \lambda)(5 + \lambda)}{7(-3 + \lambda)(-2 + \lambda)(2 + \lambda)(3 + \lambda)}.\end{aligned}\tag{3.13}$$

Then the OPE (3.2) under the large (N, k) limit can be obtained by substituting (3.12) and (3.13) into (3.2). See also the related works [32, 33] where the large (N, k) 't Hooft limit on the OPE was described.

Recently, the asymptotic symmetry of the truncated higher spin gravity in the context of black hole [34] turns out to be the classical $\mathcal{W}(2, 4)$ algebra. By changing the Poisson bracket into the commutators between Fourier modes, one obtains the various commutation relations. Note that the Fourier mode of normal ordered product field is defined as the sum of product of each Fourier mode in [16]. On the other hand, one can take the classical $c \rightarrow \infty$ limit for the OPE in (3.2). Any composite field of order n can contain only $\frac{1}{c^{n-1}}$ -behavior term. For example, the $\partial^2 T(w)$ term appears in the fourth-order pole. The c -independent term survives while c -dependent term goes away because it has $\frac{1}{c}$ behavior. The quasi-primary field containing $T^3(w)$ term in the second-order pole has cubic term, quadratic term and linear term in $T(w)$. The overall factor behaves as $\frac{1}{c^2}$ under the large c limit. Therefore, the only cubic term can survive. See also [35, 32] where the similar limiting procedure was done. Eventually, one can check the classical version of (3.2) matches with the one in [34] by turning off the structure constant C_{44}^6 which appears in front of the spin-6 current on the right hand side of OPE ⁸.

In other words, our OPE (3.2), at the quantum level, provides the asymptotic symmetry algebra in the bulk theory. The more general analysis in higher spin AdS_3 gravity corresponding to $WD_{\frac{N}{2}}$ coset minimal model should produce the spin-6 current on the right hand side of OPE and the quantum behavior coming from the normal ordering in the composite fields should appear. Note that the OPE (3.2) holds for any N and is an exact (and complete)

⁸For the WD_4 algebra, the structure constant (3.10) vanishes at $c = 1$ or $c = -\frac{656}{11}$. In [23, 27], the $\mathcal{W}(2, 4, 4)$ algebra has been shown to be consistent with for these values $c = 1$ and $c = -\frac{656}{11}$. As we take $c \rightarrow \infty$ limit with fixed N , the structure constant behaves as $(C_{44}^6)^2 \rightarrow \frac{150(-6+N)(-5+N)(3+N)(4+N)}{7(-4+N)(-3+N)(1+N)(2+N)}$ which is the ratio of each c^6 term in the denominator and numerator of $(C_{44}^6)^2$. Therefore, for $N = 6$, this structure constant vanishes. One sees the behavior of the structure constant C_{44}^4 in the classical limit. According to the observation of footnote 6, one has $(C_{44}^4)^2 \rightarrow \frac{27}{35}$ by substituting $N = 6$ into the formula. Note that this numerical value $\frac{27}{35}$ is exactly the same as the one in the classical $\mathcal{W}(2, 4)$ algebra because $\frac{54(c+24)(c^2-172c+1296)}{(5c+22)(2c-1)(7c+68)} \rightarrow \frac{54}{5 \cdot 2 \cdot 7} = \frac{27}{35}$. The WD_3 algebra reduces to the $\mathcal{W}(2, 4)$ algebra [34]. For $N = 5$, the above structure constant C_{44}^6 vanishes and from the footnote 10, the structure constant $(C_{44}^4)^2$ reduces to $\frac{27}{35}$. We thank the referee for raising this issue.

expression. Each higher spin current is made of Casimir operators that are constructed from the WZW currents. The spin-4 current has two undetermined coefficients and the complete form for the spin-6 current is not known.

4 The OPE between the spin-4 current and itself in $WB_{\frac{N-1}{2}}$ minimal model with odd N

In this case, the spin-1 field is realized by N free fermions [19, 36, 37]

$$J^{ab}(z) = \psi^a \psi^b(z). \quad (4.1)$$

The OPE between the fermions is given by

$$\psi^a(z) \psi^b(w) = \frac{1}{(z-w)} \delta^{ab} + \dots. \quad (4.2)$$

One can easily see the OPE (2.2) by using (4.1) and (4.2). One takes the other OPE (2.3). The Sugawara stress energy tensor is given by (2.5) with diagonal current (2.4). The OPE satisfies (2.6) with the central charge (2.7).

The spin-4 current in [8] is, with two undetermined coefficient functions, given by

$$\begin{aligned} V(z) = & c_3 J^{cd} J^{ef} K^{cd} K^{ef}(z) + c_9 J^{cd} K^{ef} K^{ce} K^{df}(z) + c_{10} K^{cd} K^{ef} K^{ce} K^{df}(z) \\ & + c_{11} J^{cd} J^{cd} J^{ef} J^{ef}(z) + c_{12} J^{cd} J^{cd} J^{ef} K^{ef}(z) + c_{13} J^{cd} J^{cd} K^{ef} K^{ef}(z) + c_{14} J^{cd} K^{cd} K^{ef} K^{ef}(z) \\ & + c_{15} K^{cd} K^{cd} K^{ef} K^{ef}(z) + c_{18} J^{cd} J^{ce} K^{ef} K^{df}(z) + d_1 \partial J^{ab} \partial J^{ab}(z) + d_2 \partial^2 J^{ab} J^{ab}(z) \\ & + d_3 \partial K^{ab} \partial K^{ab}(z) + d_4 \partial^2 K^{ab} K^{ab}(z) + d_5 \partial^2 J^{ab} K^{ab}(z) + d_6 \partial J^{ab} \partial K^{ab}(z) + d_7 J^{ab} \partial^2 K^{ab}(z) \\ & + d_8 J^{ab} \partial J^{ac} K^{bc}(z) + d_9 J^{ab} K^{ac} \partial K^{bc}(z), \end{aligned} \quad (4.3)$$

where the coefficient functions are given by (B.1).

Now one can compute the OPE $V(z) V(w)$ and it turns out that one has the equation (3.2). The highest singular term with $N = 7$ has

$$\begin{aligned} \text{pole } 8 &= \frac{n}{d}, \\ n &\equiv 21k \left(720c_9^2(-1+k)(2+k)^2(3+2k)^2(10+3k)(21+4k)(285+31k)^2 \right. \\ &\quad + d_8^2(5+k)(1320+79k(11+k)) \\ &\quad \times (-898722+k(201615+k(578098+k(126529+92k(107+3k))))), \\ d &\equiv (2+k)(5+k)(3+2k)(285+31k)^2(-250+23k(5+6k)), \end{aligned} \quad (4.4)$$

where there exist two unknown coefficients c_9 and d_8 . Normalizing this (4.4) to be $\frac{c}{4}$ where the central charge is

$$c_{N=7} = \frac{7k(11+k)}{2(5+k)(6+k)}, \quad (4.5)$$

one has one relation between the coefficients. The seventh order pole vanishes as before. The sixth order pole can be written in terms of $2T(w)$ where $T(w)$ is given by (2.5) with $N = 7$.

Then the unknown two coefficients are determined as

$$\begin{aligned} c_9^2 &= \frac{1320 + 869k + 79k^2}{24(2+k)(6+k)^2(9+k)(3+2k)(19+2k)(10+3k)(23+3k)}, \\ d_8^2 &= \frac{(2+k)(3+2k)(10+3k)(285+31k)^2}{24(6+k)^2(9+k)(19+2k)(23+3k)(1320+869k+79k^2)}. \end{aligned} \quad (4.6)$$

One can write down these (4.6) in terms of (4.5) but the expressions are rather complicated.

From the fourth-order pole, one obtains the self-coupling constant for the spin-4 current

$$\begin{aligned} (C_{44}^4)^2 &= \frac{150(7224 + 6677k + 2180k^2 + 286k^3 + 13k^4)^2}{(2+k)(9+k)(3+2k)(19+2k)(10+3k)(23+3k)(1320+869k+79k^2)} \\ &= \frac{2(4214 + 627c + 34c^2)^2}{(21+4c)(22+5c)(19+6c)(161+8c)}. \end{aligned} \quad (4.7)$$

This coincides with the results [38] from the quantum Miura transformation. It is easy to check that one also obtains (4.7) from (3.8) by putting $N = 7$. The classical $c \rightarrow \infty$ limit of (3.8) coincides with the one in [29] where the structure constant can be obtained from the one from WA_{N-1} minimal model ⁹.

Also one can read off the spin-6 current

$$C_{44}^6 W(z) = \text{coeff}(k) \psi^a \partial^5 \psi^a(z) + \dots, \quad (4.8)$$

where $\text{coeff}(k)$ is a complicated function of k . One also expects that one obtains the following structure constant, after computing the OPE between the spin-6 current (4.8) and itself,

$$\begin{aligned} (C_{44}^6)^2 &= \frac{n}{d} = \frac{80(-1+c)(49+c)^2(22+5c)^2(403+22c)}{3(24+c)(-1+2c)(21+4c)(19+6c)(68+7c)(161+8c)}, \\ n &\equiv 5(-1+k)(4+k)^2(7+k)^2(12+k)(13+4k)(31+4k)(1320+869k+79k^2)^2, \\ d &\equiv (2+k)(9+k)(3+2k)(19+2k)(10+3k)(23+3k)(-5+11k+k^2) \\ &\quad \times (288+121k+11k^2)(816+407k+37k^2). \end{aligned} \quad (4.9)$$

⁹The self-coupling structure constant between the spin-4 currents in the WA_{N-1} minimal model is given by [30] $(C_{44}^4)^2 = \frac{36(82+45c-19c^2-94N^2-75cN^2+c^2N^2+12N^3+18cN^3)^2}{(2+c)(22+5c)(-3+N)(-2+N)(-2+2c-N+cN+3N^2)(-6+3c-2N+cN+8N^2)}$. By taking the $c \rightarrow \infty$ limit, this leads to $\frac{36(-19+N^2)^2}{5(-3+N)(-2+N)(2+N)(3+N)}$ which is equal to the corresponding limit of (3.8) with N replaced by $N+1$ [29].

One sees this expression from (3.11) by taking $N = 7$. Also this structure constant (4.9) appeared in [38]. The lowest OPE between the spin-4 current and itself in the $WB_{\frac{N-1}{2}}$ coset minimal model is characterized by (3.2) where the central charge is given by (2.7), the spin-4 current is given by (4.3), the spin-2 current is given by (2.5), the structure constant C_{44}^4 is given by (3.8), and the structure constant C_{44}^6 is given by (3.11). For the spin-4 and spin-6 currents, we have found for particular $N = 7$ case (and $N = 5$ case) ¹⁰. As described before, when $N = 3$ (i.e., WB_1 coset minimal model), the structure constants (3.8) and (3.11) produce the previous known results in [27] ¹¹. As before, the ‘minimal’ extension of conformal algebra arises for $k = 1$ where the only higher spin current is of spin-4 while the higher spin current of spin-6 vanishes.

5 Conclusions and outlook

We have found the OPEs (3.2) between the spin-4 current and itself in the $WD_{\frac{N}{2}}$ and $WB_{\frac{N-1}{2}}$ coset minimal models, by checking those in WD_3 and WD_4 minimal models (and WB_2 and WB_3) explicitly. These are the simplest OPEs for given minimal models. By using the holography between the above conformal field theory and higher spin AdS_3 gravity, we expect that the bulk computation, at the quantum level, should possess the asymptotic symmetry corresponding to the OPE (3.2).

It is an open problem to find the correct answer for the following interesting subjects.

- The full expressions of the spin-4 and the spin-6 currents at general N . So far, the spin-4 current is found, for general N , up to two unknown coefficients. This can be done only after one should compute the OPE between the spin-4 current and itself by hand. After doing this complicated long computation, one can extract the spin-4 current and the spin-6 current (up to an overall factor) completely. Or one can follow the method in [8] in order to obtain the spin-6 current by imposing that the OPE with diagonal current has no singular term and the spin-6 current is primary field under the stress energy tensor. It is nontrivial to exhaust all the possible terms coming from sextic-, \dots , cubic- and quadratic-terms in WZW currents.

- The quantum Miura transformations and the corresponding higher spin currents. One can also find the higher spin currents using the quantum Miura transformation. Then it

¹⁰ We have also checked for the OPEs in WB_2 minimal model [39, 37] where the spin-6 current contains a term $U\partial U(w)$ and $U(w)$ is a spin- $\frac{5}{2}$ current [8]. The structure constants $(C_{44}^4)^2 = \frac{54(-490+83c+2c^2)^2}{(25+2c)^2(22+5c)(13+14c)}$ and $(C_{44}^6)^2 = \frac{720(-1+c)(115+4c)(22+5c)^2(49+6c)}{(24+c)(-1+2c)(25+2c)^2(68+7c)(13+14c)}$ in WB_2 minimal model can be obtained from (3.8) and (3.11) by plugging $N = 5$ respectively. There is a $(c-1)$ factor in C_{44}^6 .

¹¹ That is, $(C_{44}^4)^2 = \frac{54(-82+47c+10c^2)^2}{(21+4c)(22+5c)(-7+10c)}$ and $(C_{44}^6)^2 = \frac{144(-1+c)^2(11+c)(22+5c)^2(11+14c)}{(24+c)(-1+2c)(21+4c)(68+7c)(-7+10c)}$. There is a $(c-1)$ factor in the second structure constant.

is straightforward to compute the OPE between the spin-4 and the spin-6 and the OPE between the spin-6 and itself. The nontrivial part in this direction is to obtain the primary fields under the stress energy tensor by combining the nonprimary fields with fixed spins. Also it is interesting to see whether the other structure constants in [11] are correct or not.

- Any supersymmetric extensions? So far, the supersymmetric versions of minimal model holography are described in the recent works [40, 41, 42, 43, 32, 44, 33, 45, 46, 47, 48, 49, 50]. In particular, it would be interesting to see whether the $WB_{\frac{N-1}{2}}$ minimal model can be generalized to the supersymmetric extension or not. In the coset model (2.1), one of the level is fixed by one. What happens if this level is given by N along the line of [17]? As pointed out in [12], it is an open problem to find out other supersymmetric coset minimal models.

- It is natural to ask whether the next higher spin-5 Casimir current in the context of [6] can be obtained from the OPE between the spin-3 current and the spin-4 current or not. It would be interesting to construct the spin-5 current explicitly.

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Appendix A The coefficients in the spin-4 current of $WD_{\frac{N}{2}}$ minimal model

The explicit coefficient functions [8] in (3.1), in terms of c_8 and c_{10} , are given by

$$\begin{aligned}
c_3 &= \left(c_8 \left(k^2(44 + 5N) + 44(2 - 3N + N^2) + k(-132 + 73N + 10N^2) \right) \right. \\
&\quad \left. - 2c_{10} \left(42k^4 + k^3(-338 + 163N) + 2(-2 + N)^2(76 - 67N + 12N^2) \right. \right. \\
&\quad \left. \left. + k^2(1000 - 953N + 219N^2) + k(-1288 + 1826N - 835N^2 + 122N^3) \right) \right) / \\
&\quad \left(k^2(44 + 5N) + 44(2 - 3N + N^2) + k(-132 + 73N + 10N^2) \right), \\
c_9 &= -4c_{10}(-2 + k + N), \\
c_{11} &= - \left(k \left(-c_8(-1 + k) \left(5 - 3N + N^2 \right) \left(k^2(44 + 5N) + 44(2 - 3N + N^2) \right. \right. \right. \\
&\quad \left. \left. + k(-132 + 73N + 10N^2) \right) + 2c_{10} \left(44(-2 + N)^3(4 - 5N + N^2) \right. \right. \\
&\quad \left. \left. + 6k^5(29 - 15N + 7N^2) + k^4(-1438 + 1433N - 683N^2 + 163N^3) \right. \right. \\
&\quad \left. \left. + 2k(-2 + N)^2(648 - 876N + 371N^2 - 92N^3 + 12N^4) \right. \right. \\
&\quad \left. \left. + 3k^3(1536 - 2249N + 1377N^2 - 471N^3 + 73N^4) \right. \right. \\
&\quad \left. \left. + k^2(-7096 + 13610N - 10495N^2 + 4424N^3 - 1090N^4 + 122N^5) \right) \right) / \\
&\quad \left(2(-1 + N)^2(2 - 5N + 2N^2) \left(k^2(44 + 5N) + 44(2 - 3N + N^2) \right. \right. \\
&\quad \left. \left. + k(-132 + 73N + 10N^2) \right) \right), \\
c_{12} &= \left(2 \left(-c_8(-1 + k) \left(5 - 3N + N^2 \right) \left(k^2(44 + 5N) + 44(2 - 3N + N^2) \right. \right. \right. \\
&\quad \left. \left. + k(-132 + 73N + 10N^2) \right) + 2c_{10} \left(44(-2 + N)^3(4 - 5N + N^2) \right. \right. \\
&\quad \left. \left. + 6k^5(29 - 15N + 7N^2) + k^4(-1438 + 1433N - 683N^2 + 163N^3) \right. \right. \\
&\quad \left. \left. + 2k(-2 + N)^2(648 - 876N + 371N^2 - 92N^3 + 12N^4) \right. \right. \\
&\quad \left. \left. + 3k^3(1536 - 2249N + 1377N^2 - 471N^3 + 73N^4) \right. \right. \\
&\quad \left. \left. + k^2(-7096 + 13610N - 10495N^2 + 4424N^3 - 1090N^4 + 122N^5) \right) \right) / \\
&\quad \left((-2 + 7N - 7N^2 + 2N^3) \left(k^2(44 + 5N) + 44(2 - 3N + N^2) \right. \right. \\
&\quad \left. \left. + k(-132 + 73N + 10N^2) \right) \right), \\
c_{13} &= - \left(3 \left(-c_8 \left(k^2(44 + 5N) + 44(2 - 3N + N^2) + k(-132 + 73N + 10N^2) \right) \right. \right. \\
&\quad \left. \left. + 2c_{10} \left(44(-2 + N)^3(-1 + N) + 6k^4(5 + 2N) + 2k(-2 + N)^2(-101 + 73N + 7N^2) \right. \right. \right. \\
&\quad \left. \left. \left. + k^3(-238 + 64N + 41N^2) + k^2(672 - 656N + 74N^2 + 43N^3) \right) \right) \right) / \\
&\quad \left((2 - 3N + N^2) \left(k^2(44 + 5N) + 44(2 - 3N + N^2) + k(-132 + 73N + 10N^2) \right) \right),
\end{aligned}$$

$$\begin{aligned}
c_{14} &= \frac{12c_{10}(-2+k+N)(18+7k^2-15N+4N^2+7k(-3+2N))}{k^2(44+5N)+44(2-3N+N^2)+k(-132+73N+10N^2)}, \\
c_{15} &= -\frac{3c_{10}(18+7k^2-15N+4N^2+7k(-3+2N))}{k^2(44+5N)+44(2-3N+N^2)+k(-132+73N+10N^2)}, \\
c_{18} &= \frac{-6c_8+4c_{10}(3k^2+5k(-2+N)+2(-2+N)^2)}{-2+N}, \\
c_{21} &= \left(k \left(c_8(-1+k)(-8+N) + 2c_{10} \left(6k^3 + 2(-4+N)(-2+N)^2 \right. \right. \right. \\
&\quad \left. \left. \left. + k^2(-32+13N) + k(56-46N+9N^2) \right) \right) \right) / \left(2(-2+7N-7N^2+2N^3) \right), \\
c_{22} &= -\frac{1}{2-5N+2N^2} 2 \left(c_8(-1+k)(-8+N) \right. \\
&\quad \left. + 2c_{10} \left(6k^3 + 2(-4+N)(-2+N)^2 + k^2(-32+13N) + k(56-46N+9N^2) \right) \right), \\
d_1 &= \left(k \left(-c_8(-1+k)(-2+N) \left(k^2(44+5N) + 44(2-3N+N^2) \right. \right. \right. \\
&\quad \left. \left. \left. + k(-132+73N+10N^2) \right) \right) + 2c_{10} \left(6k^5(-8+N) + k^4(472-322N+25N^2) \right. \right. \\
&\quad \left. \left. - 8(-2+N)^2(-28+55N-32N^2+5N^3) + k^3(-1768+2172N-705N^2+35N^3) \right. \right. \\
&\quad \left. \left. + k^2(3168-5644N+3294N^2-663N^3+20N^4) \right. \right. \\
&\quad \left. \left. + k(-2720+6408N-5460N^2+2000N^3-274N^4+4N^5) \right) \right) / \\
&\quad \left(4(-1+N) \left(k^2(44+5N) + 44(2-3N+N^2) + k(-132+73N+10N^2) \right) \right), \\
d_2 &= \left(k \left(c_8(-1+k)(-4+N^2) \left(k^2(44+5N) + 44(2-3N+N^2) \right. \right. \right. \\
&\quad \left. \left. \left. + k(-132+73N+10N^2) \right) \right) - 2c_{10} \left(18k^5(-8+N^2) \right. \right. \\
&\quad \left. \left. + 3k^4(384-152N-82N^2+25N^3) \right. \right. \\
&\quad \left. \left. - 8(-2+N)^2(-40+58N-8N^2-13N^3+3N^4) \right. \right. \\
&\quad \left. \left. + k^3(-3632+2992N+464N^2-727N^3+105N^4) \right. \right. \\
&\quad \left. \left. + k^2(5632-7304N+1064N^2+1778N^3-705N^4+60N^5) \right. \right. \\
&\quad \left. \left. + 2k(-2144+3928N-1764N^2-552N^3+590N^4-127N^5+6N^6) \right) \right) / \\
&\quad \left(4(1-3N+2N^2) \left(k^2(44+5N) + 44(2-3N+N^2) + k(-132+73N+10N^2) \right) \right), \\
d_3 &= \left(3c_{10} \left(2k^4(-8+N) + k^3(96-76N+8N^2) + k^2(-104+216N-103N^2+12N^3) \right. \right. \\
&\quad \left. \left. + 4(36-84N+71N^2-27N^3+4N^4) + k(-120+62N+57N^2-42N^3+8N^4) \right) \right) / \\
&\quad \left(2 \left(k^2(44+5N) + 44(2-3N+N^2) + k(-132+73N+10N^2) \right) \right), \\
d_4 &= - \left(c_{10} \left(2k^4(-8+N) + k^3(96-76N+8N^2) + k^2(-104+216N-103N^2+12N^3) \right. \right. \\
&\quad \left. \left. + 4(36-84N+71N^2-27N^3+4N^4) + k(-120+62N+57N^2-42N^3+8N^4) \right) \right) / \\
&\quad \left(k^2(44+5N) + 44(2-3N+N^2) + k(-132+73N+10N^2) \right),
\end{aligned}$$

$$\begin{aligned}
d_5 &= \left(-6c_8(-1+k)(-3+N)N \left(k^2(44+5N) + 44(2-3N+N^2) \right. \right. \\
&\quad \left. \left. + k(-132+73N+10N^2) \right) + c_{10} \left(6k^5N(-197+15N+14N^2) \right. \right. \\
&\quad \left. \left. + k^4(96+9502N-5303N^2-445N^3+350N^4) \right. \right. \\
&\quad \left. \left. + 4(-2+N)^2(-64+1060N-1945N^2+1237N^3-316N^4+28N^5) \right. \right. \\
&\quad \left. \left. + k^3(-320-31012N+33947N^2-8004N^3-1485N^4+490N^5) \right. \right. \\
&\quad \left. \left. + k^2(-128+52536N-87576N^2+47653N^3-7523N^4-1042N^5+280N^6) \right. \right. \\
&\quad \left. \left. + 2k(640-23560N+52290N^2-44415N^3+17202N^4-2685N^5-4N^6+28N^7) \right) \right) / \\
&\quad \left((1-3N+2N^2) \left(k^2(44+5N) + 44(2-3N+N^2) + k(-132+73N+10N^2) \right) \right), \\
d_6 &= - \left(5c_8(-2+N) \left(k^2(44+5N) + 44(2-3N+N^2) + k(-132+73N+10N^2) \right) \right. \\
&\quad \left. + 4c_{10} \left(3k^5(-8+N) + k^4(368-212N+5N^2) - k^3(1720-1893N+492N^2+10N^3) \right. \right. \\
&\quad \left. \left. - 4(-2+N)^2(-72+143N-87N^2+16N^3) \right. \right. \\
&\quad \left. \left. + k^2(3520-5888N+3141N^2-494N^3-20N^4) \right. \right. \\
&\quad \left. \left. - 2k(1648-3786N+3149N^2-1110N^3+130N^4+4N^5) \right) \right) / \\
&\quad \left(2 \left(k^2(44+5N) + 44(2-3N+N^2) + k(-132+73N+10N^2) \right) \right), \\
d_7 &= \left(c_{10}(-2+k+N) \left(2k^4(-8+N) + k^3(176-126N+13N^2) \right. \right. \\
&\quad \left. \left. + k^2(-416+584N-249N^2+32N^3) + 4(-48-40N+130N^2-77N^3+14N^4) \right. \right. \\
&\quad \left. \left. + 2k(176-240N+161N^2-69N^3+14N^4) \right) \right) / \\
&\quad \left(k^2(44+5N) + 44(2-3N+N^2) + k(-132+73N+10N^2) \right), \\
d_8 &= \left(2 \left(-c_8(-1+k) \left(-10-21N+7N^2 \right) \left(k^2(44+5N) + 44(2-3N+N^2) \right. \right. \right. \\
&\quad \left. \left. + k(-132+73N+10N^2) \right) + 4c_{10} \left(6k^5(-28-62N+N^2+5N^3) \right. \right. \\
&\quad \left. \left. + 2(-2+N)^3(-76-233N+471N^2-182N^3+20N^4) \right. \right. \\
&\quad \left. \left. + k^4(1436+2274N-1474N^2-261N^3+125N^4) \right. \right. \\
&\quad \left. \left. + k(-2+N)^2(-1252-2285N+4489N^2-1538N^3+62N^4+20N^5) \right. \right. \\
&\quad \left. \left. + k^3(-4676-5144N+9067N^2-1946N^3-668N^4+175N^5) \right. \right. \\
&\quad \left. \left. + k^2(7152+5758N-22139N^2+13624N^3-1946N^4-449N^5+100N^6) \right) \right) / \\
&\quad \left((-2+7N-7N^2+2N^3) \left(k^2(44+5N) + 44(2-3N+N^2) \right. \right. \\
&\quad \left. \left. + k(-132+73N+10N^2) \right) \right), \\
d_9 &= \left(4c_{10}(-2+k+N) \left(5k^3(-8+N) + 2k^2(78-53N+10N^2) \right. \right. \\
&\quad \left. \left. + 4(42-N-30N^2+10N^3) + k(-236+153N-56N^2+20N^3) \right) \right) /
\end{aligned}$$

$$\left(k^2(44 + 5N) + 44(2 - 3N + N^2) + k(-132 + 73N + 10N^2)\right). \quad (\text{A.1})$$

The coefficients c_8 and c_{10} for $N = 8$ are determined by (3.7) and (3.5). We also obtained those coefficients for $N = 6$. For general N , they are not known so far.

Appendix B The coefficients in the spin-4 current of $WB_{\frac{N-1}{2}}$ minimal model

The explicit coefficient functions [8] in (4.3), in terms of c_9 and d_8 , are given by

$$\begin{aligned} c_3 &= -\frac{d_8(-8 + N)(-19 + 7k + 12N)}{2(2 + k)(68 - 39N + 10N^2 + k(-4 + 5N))}, \\ c_{10} &= -\frac{c_9}{4(-2 + k + N)}, \\ c_{11} &= -\frac{d_8k(6k + 11(-2 + N))}{4(-1 + N)(68 - 39N + 10N^2 + k(-4 + 5N))}, \\ c_{12} &= \frac{d_8(6k + 11(-2 + N))}{68 - 39N + 10N^2 + k(-4 + 5N)}, \\ c_{13} &= -\frac{3d_8(-8 + N)(2k^2(5 + 2N) + 22(2 - 3N + N^2) + k(-46 + 18N + 7N^2))}{2(2 + k)(-4 + 2k + N)(2 - 3N + N^2)(68 - 39N + 10N^2 + k(-4 + 5N))}, \\ c_{14} &= \left(d_8(-8 + N)^2(-2 + k + N)(-16 + 11N) \right. \\ &\quad + c_9(-2992 + 4164N - 2456N^2 + 779N^3 - 129N^4 + 10N^5 \\ &\quad - 6k^3(4 - 13N + 10N^2) + k^2(416 - 1080N + 561N^2 - 130N^3) \\ &\quad \left. + k(40 + 54N - 135N^2 + 79N^3 - 15N^4)\right) / \left((68 - 39N + 10N^2 + k(-4 + 5N)) \right. \\ &\quad \left. (-20(-2 + N)^2 + 6k^2(4 - N + N^2) + k(-8 + 6N - 3N^2 + N^3))\right), \\ c_{15} &= \left(-d_8(-8 + N)^2(-2 + k + N)(-16 + 11N) \right. \\ &\quad + c_9(2992 - 4164N + 2456N^2 - 779N^3 + 129N^4 - 10N^5 + 6k^3(4 - 13N + 10N^2) \\ &\quad + k^2(-416 + 1080N - 561N^2 + 130N^3) + k(-40 - 54N + 135N^2 - 79N^3 + 15N^4)) \left. \right) / \\ &\quad \left(4(-2 + k + N)(68 - 39N + 10N^2 + k(-4 + 5N))(-20(-2 + N)^2 \right. \\ &\quad \left. + 6k^2(4 - N + N^2) + k(-8 + 6N - 3N^2 + N^3))\right), \\ c_{18} &= \frac{d_8(-8 + N)(k^2(44 + 5N) + 44(2 - 3N + N^2) + k(-132 + 73N + 10N^2))}{(2 + k)(-2 + N)(-4 + 2k + N)(68 - 39N + 10N^2 + k(-4 + 5N))}, \\ d_1 &= \frac{3d_8k(-4 + N + 2N^2 + k(4 + N))}{4(68 - 39N + 10N^2 + k(-4 + 5N))}, \\ d_2 &= -\frac{d_8k(-4 + N + 2N^2 + k(4 + N))}{2(68 - 39N + 10N^2 + k(-4 + 5N))}, \end{aligned}$$

$$\begin{aligned}
d_3 &= - \left(3 \left(-2d_8(-8+N)^2 \left(4(-2+N)^2(-1+N) + k^3(4-N+N^2) \right) \right. \right. \\
&\quad + k^2(-20+17N-8N^2+3N^3) + k(32-46N+25N^2-9N^3+2N^4) \Big) \\
&\quad + c_9(-2+N) \left(7616-13072N+9648N^2-3852N^3+832N^4-80N^5 \right. \\
&\quad + 2k^3(48-116N+82N^2-19N^3+5N^4) \\
&\quad + k(-992-656N+854N^2-291N^3+11N^4+10N^5) \\
&\quad + k^2(-1600+2920N-1926N^2+691N^3-133N^4+20N^5) \Big) \Big) / \\
&\quad \left(8(-136+146N-59N^2+10N^3+k^2(-4+5N)+k(76-53N+15N^2)) \right. \\
&\quad \left. (-20(-2+N)^2+6k^2(4-N+N^2)+k(-8+6N-3N^2+N^3)) \right), \\
d_4 &= \left(-2d_8(-8+N)^2 \left(4(-2+N)^2(-1+N) + k^3(4-N+N^2) \right) \right. \\
&\quad + k^2(-20+17N-8N^2+3N^3) + k(32-46N+25N^2-9N^3+2N^4) \Big) \\
&\quad + c_9(-2+N) \left(7616-13072N+9648N^2-3852N^3+832N^4-80N^5 \right. \\
&\quad + 2k^3(48-116N+82N^2-19N^3+5N^4) \\
&\quad + k(-992-656N+854N^2-291N^3+11N^4+10N^5) \\
&\quad + k^2(-1600+2920N-1926N^2+691N^3-133N^4+20N^5) \Big) \Big) / \\
&\quad \left(4(-136+146N-59N^2+10N^3+k^2(-4+5N)+k(76-53N+15N^2)) \right. \\
&\quad \left. (-20(-2+N)^2+6k^2(4-N+N^2)+k(-8+6N-3N^2+N^3)) \right), \\
d_5 &= \frac{d_8(-128-8(-17+k)N+(-61+7k)N^2+14N^3)}{4(68-39N+10N^2+k(-4+5N))}, \\
d_6 &= \left(d_8(-8+N) \left(-k^3(-8+N) + 6k^2(-16+9N) + k(232-267N+66N^2+4N^3) \right) \right. \\
&\quad + 2(-72+143N-87N^2+16N^3) \Big) \Big) / (2(2+k)(-4+2k+N) \\
&\quad (68-39N+10N^2+k(-4+5N))), \\
d_7 &= \left(d_8(-8+N)^2 \left(4(-2+N)^2(-23+18N) + 2k^3(4-N+N^2) \right) \right. \\
&\quad + k^2(-80+64N-31N^2+11N^3) + k(312-414N+199N^2-63N^3+14N^4) \Big) \\
&\quad - 4c_9(-2+N) \left(7616-13072N+9648N^2-3852N^3+832N^4-80N^5 \right. \\
&\quad + 2k^3(48-116N+82N^2-19N^3+5N^4) \\
&\quad + k(-992-656N+854N^2-291N^3+11N^4+10N^5) \\
&\quad + k^2(-1600+2920N-1926N^2+691N^3-133N^4+20N^5) \Big) \Big) / \\
&\quad \left(4(68-39N+10N^2+k(-4+5N)) \left(-20(-2+N)^2+6k^2(4-N+N^2) \right) \right. \\
&\quad + k(-8+6N-3N^2+N^3) \Big) \Big),
\end{aligned}$$

$$\begin{aligned}
d_9 = & \left(d_8(-8 + N)^2 \left(5k^2(4 - N + N^2) + 4(42 - 53N + 16N^2) \right. \right. \\
& + k(-124 + 99N - 25N^2 + 10N^3) \Big) \\
& - 3c_9(7616 - 13072N + 9648N^2 - 3852N^3 + 832N^4 - 80N^5 \\
& + 2k^3(48 - 116N + 82N^2 - 19N^3 + 5N^4) \\
& + k(-992 - 656N + 854N^2 - 291N^3 + 11N^4 + 10N^5) \\
& + k^2(-1600 + 2920N - 1926N^2 + 691N^3 - 133N^4 + 20N^5) \Big) \Big) / \\
& \left((68 - 39N + 10N^2 + k(-4 + 5N))(-20(-2 + N)^2 + 6k^2(4 - N + N^2) \right. \\
& + k(-8 + 6N - 3N^2 + N^3) \Big) \Big). \tag{B.1}
\end{aligned}$$

The coefficients c_9 and d_8 for $N = 7$ are determined by (4.6). For general N , they are not known so far. We also obtained those coefficients for $N = 5$.

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